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# IMPROVED MODEL OF THE GRIFFITH CRACK

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*The plane-stress state of a cracked continuous medium in tension is determined using relaxation elements. The stress state is analyzed at the tip of a crack surrounded by a plastically deformed material as a band of localized plastic deformation* (LPD) *shaped like an elongated ellipse. The plastic deformation considerably decreases the stress concentration at the crack tip. As the localization of the plastic deformation increases, the stresses at the crack sides decrease to zero. The decrease in stresses at the tip is accompanied by an increase in the concentration and gradients of the stresses at the end of the* LPD *band. Here the region of perturbation of the stress field is comparable with the width of the band.*

**Introduction.** The principles and criteria of linear fracture mechanics [1–5] usually underlie the strength and durability calculations for the structural members. The criteria are calculated with allowance for the properties of the classical model of the Griffith crack despite its having significant two drawbacks. The first is the presence of a singular point at the end of a cut, an unbounded increase in the stress being observed as this point is approached. This explains the fact that the physical notion of the coefficient of stress concentration at the crack tip is not used in fracture mechanics, and the coefficient of stress intensity in the neighborhood of the singular point is used as the characteristic of the inhomogeneous stress field. To formulate the criterion for crack propagation, Griffith assumed that the formation of the crack surface is connected with the expenditure of energy. As the crack lengthens, the released elastic energy should be higher than the energy expended for the formation of the new surfaces. The additional energy $\gamma$ introduced by Griffith cannot be calculated from the elasticity equations for a solid body with a cut, which may be considered as the second drawback of the Griffith theory.

The assumptions of the nonlinear behavior of a material at the crack ends allowed one to employ a number of known models for calculations [2–6]. However, their application is restricted. They hold true in the case where the plastic-deformation zone is very small compared with the length of the crack. The description of the stress state near the crack tip with an extended plastic-deformation zone involves great mathematical and computing difficulties [2–5].

This work suggests an improved model of the Griffith crack that is free from these drawbacks owing to the assumption that the effective moduli of elasticity in the immediate proximity of the real surface of a solid body differ from those in the volume of the material and owing to taking into account the relaxation effects (the stress decrease) in the plastic-deformation zone. This allowed us to characterize the crack in a solid body as a defect with the internal-stress field and to relate the formation of the free surface with work of the external forces.

The method of relaxation elements was used to construct the model of a crack with the site of plastic deformation in a continuous medium and to calculate the stress state of this medium [7–9]. For the plane-stress state, analytical expressions were obtained which are applicable in engineering calculations of the stress concentration and in the formulation of the fracture criteria for cracked materials containing the plastic-deformation zones.

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**1. Taking into Account the Physical Width of the Crack Surface.** We shall consider an opening crack oriented perpendicularly to the axis of tension in the form of a cut in an isotropic continuous medium. We present the free surface of the cut as a layer of gradient material within which the Young modulus continuously varies from zero to $E_0$ in the volume of the material. The thickness of the surface layer is called the physical width of the crack surface. This thickness should be much less than the length of the crack, because the number of links between the atoms and, hence, the forces of interaction between them in an actual surface layer weaken toward the external atomic layer [10]. One can model this situation by representing the surface as a thin layer of a continuous material in the neighborhood of which the elastic moduli continuously tend to zero as the surface frontier is approached.

It is known that the decrease in the elastic modulus in the local continuous-medium region under external loading decreases the stress in this region and increases the stress concentration outside this region in the neighborhood of its boundary. If there is a local region shaped like an ellipse with Young modulus $E_0$ and with center at the origin of the Cartesian coordinate system in an elastically deformable plate with Young modulus $E_1$, it is possible to show that inside the ellipse, in the tension by the force $\sigma$ along its minor semi-axis $b$ directed along the $y$ axis the stress is $\sigma - \Delta\sigma$, where $\Delta\sigma = \sigma(E_0 - E_1)b/(2aE_1 + bE_0)$, $E_0 > E_1$, and $a$ is the other semi-axis of the ellipse. Because of the stress decrease by $\Delta\sigma$, the additional variation in the shape of the ellipse is determined by the strain tensor with the components [7–9] $\Delta\varepsilon_x = \Delta\sigma/E_0$, $\Delta\varepsilon_y = (1+2a/b)\Delta\sigma/E_0$, and $\Delta\tau_{xy} = 0$. Specifying $E_1$ or $\Delta\sigma$ completely determines the stress-strain state of the plane subjected to the stress $\sigma$. The Young modulus $E_0$ is considered specified in the volume of the material.

There is an inhomogeneous stress field outside the elliptical region. The stress-tensor components along the $x$ axis are determined by the relations

$$\sigma_x = \Delta\sigma \frac{a^2}{(a-b)^2}\Big[\frac{x}{\sqrt{x^2-a^2+b^2}} - 1 - \frac{b^2x(1-b/a)}{(x^2-a^2+b^2)^{3/2}}\Big],$$
$$\sigma_y = \sigma - \Delta\sigma + \Delta\sigma\frac{a^2}{(a-b)^2}\Big[\frac{b^2}{a^2} + \frac{x(1-2b/a)}{\sqrt{x^2-a^2+b^2}} + \frac{b^2x(1-b/a)}{(x^2-a^2+b^2)^{3/2}}\Big], \quad \sigma_{xy} = 0. \tag{1.1}$$

We note that without the homogeneous stress field $\sigma-\Delta\sigma$ (see the expression for $\sigma_y$) Eqs. (1.1) describe the stress state of the plate with an elliptical notch under the action of the external stress $\Delta\sigma$. We arrive at the same result in considering a region filled with a material with the Young modulus tending to zero instead of an orifice ($E_1 \to 0$). Here we have $\Delta\sigma \to \sigma$, i.e., the complete relaxation of the stresses is observed. Thus, the decrease in the modulus causes the effect of stress relaxation (the stress decrease) and the appearance of an inhomogeneous stress field around the relaxation region. The influence of a smooth variation in the modulus of elasticity in the layer before the boundary of the elliptical contour on the stress state of the plane is taken into account by the method of relaxation elements [7–9].

We present the crack as an elliptical hollow with a layer over the contour within which the modulus of elasticity $E$ continuously increases from 0 to $E_0$ in the volume of the material. Figure 1 shows the ellipse-shaped relaxation elements (RE) inserted into each other. It is assumed that outside this layer the matrix is homogeneous, isotropic, and is elastically deformed at the tensile stress $\sigma$ along the $y$ axis. For definiteness, it is necessary to describe the geometrical parameters of each RE from the family. In addition, inside each RE from this family one needs to specify the magnitude of the elementary decrease in the stress (the elementary tensor of relaxation) such that the total decrease in the stresses of all the RE eliminates all the stresses in the internal region (the hollow) if the internal stress is $\sigma$.

We accomplish this as follows. We assume that all ellipses are center-sharing at the coordinate origin and have semi-axes coinciding with the axes of the coordinates. The lengths of the semi-axes (Fig. 1) are found by the equalities $a(t) = a_0 + h(1-t)$ and $b(t) = b_0 + h(1-t)$, where $h$ is the thickness of the layer, which is the same along the $x$ and $y$ axes; $t$ is the variable varying from 0 to 1. The variation of the layer thickness in other directions is ignored. The semi-axes are maximum at $t = 0$. Therefore, the increase in $t$ corresponds to a subsequent transition from the external ellipse to the boundary of the hollow. The value of $t$ corresponds to a definite contour of the family. As $b_0 \to 0$, the hollow becomes a crack. We assume that for

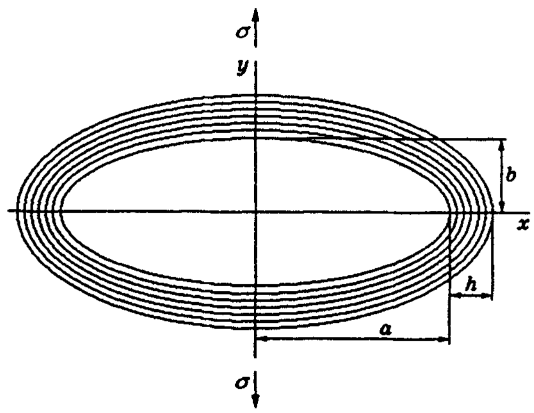


Fig. 1

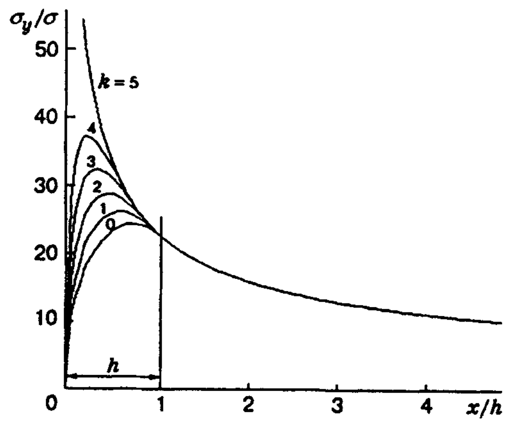


Fig. 2

the crack $a_0 \gg h$, the point on the $x$ axis corresponds to the RE contour for

$$t = 1 - (x - a_0)/h. \tag{1.2}$$

The magnitude of the elementary-relaxation tensor for each RE is expressed as a function of the variable $t$:

$$d\sigma^r = (\beta + 1)\sigma t^\beta \, dt, \qquad -1 \leqslant \beta \leqslant \infty. \tag{1.3}$$

It is seen that the parameter $\beta$ governs the relaxation in the continuous contour-to-contour transition. The larger the magnitude of $t$, the greater the elementary-relaxation tensor. The principle of superposition is true for the RE, because the elementary fields (solutions) for the stresses in the approximation of the linear theory of elasticity are summed. The normalization coefficient $\beta + 1$ ensures the absence of stresses inside the hollow, since in the integration of $d\sigma^r$ from 0 to 1 it yields a decrease in the stress equal to the stress $\sigma$ applied from outside. In the integration of $d\sigma^r$ from 0 to $t = 1 - (x - a_0)/h$, the stress decrease inside the chosen layer is the larger the closer the point to the hollow. The parameter $\beta$ governs the rate of variation of this quantity: the larger the $\beta$, the more rapidly the stress relaxes at the boundary of the hollow. Thus we have modeled the corresponding decrease in the effective moduli of elasticity toward the cut boundary.

Since there is an unambiguous relationship between the elementary-relaxation tensor inside the RE and the elementary stress field outside the RE [see Eqs. (1.1)], specifying the magnitude of the relaxation in the local regions by means of the RE distribution automatically determines the resulting inhomogeneous stress field in the entire plane, including the layer. In the adopted coordinate system (Fig. 1), for the component $\sigma_y$ of the elementary stress field of an arbitrary RE along the $x$ axis, according to conditions (1.2) and (1.3) one can write the expression

$$d\sigma_y = \sigma(\beta + 1)t^\beta \left[\frac{h^2(1-t)^2}{a_0^2} + \frac{x}{\sqrt{x^2 - a_0^2}} + \frac{h^2(1-t)^2 x}{(x^2 - a_0^2)^{3/2}}\right] dt. \tag{1.4}$$

Integrating expression (1.4) over the variable $t$ and taking into account that the integration limits are taken from 0 to 1 outside the layer and from $t$ up to 1 at the points entering the layer, where $t$ depends on the coordinate of the point, according to the definition (1.2) we obtain the equation for the profile of the component $\sigma_y$ along the $x$ axis:

$$\frac{\sigma_y}{\sigma} = \frac{2h^2}{(\beta+2)(\beta+3)}\left(\frac{1}{a_0^2} + \frac{x}{(x^2 - a_0^2)^{3/2}}\right) + \frac{x}{\sqrt{x^2 - a_0^2}} = A(x), \quad x \geqslant a_0 + h,$$

and

$$\frac{\sigma_y}{\sigma} = A(x)\Big[1 - \Big(1 - \frac{x - a_0}{h}\Big)^{\beta+1}\Big] + \frac{\beta+1}{\beta+3}h^2\Big(\frac{1}{a_0^2} + \frac{x}{(x^2 - a_0^2)^{3/2}}\Big)\Big(\frac{\beta+4}{\beta+2} + \frac{x - a_0}{h}\Big)\Big(1 - \frac{x - a_0}{h}\Big) \qquad (1.5)$$

if $a_0 \leqslant x \leqslant a_0 + h$.

Figure 2 shows the change in the distribution of the stress $\sigma_y$ in the neighborhood of the crack, according to Eq. (1.5) for $\beta = 1.5^k$ and $a_0/h = 100^3$. It is clear that, in contrast to the Griffith solution (the curve for $k = 5$), there is no singularity at the end of the crack in this case. In the near-surface layer the stress continuously increases, beginning from zero at the end of the cut, passes through the maximum, and decreases, asymptotically approaching the magnitude of the external stress $\sigma$. Outside the layer the qualitative and quantitative differences of the curves almost disappear. A decrease in the parameter $\beta$ increases the stress concentration and shifts the maximum to the boundary of the hollow. As $\beta \to \infty$, we obtain the Griffith curve in the limit. This effect occurs when the physical width of the surface $h$ decreases.

The advantages of this variant are evident, because it produces the singular solution only as a particular case where the thickness of the near-surface layer tends to zero or the parameter $\beta$ tends to infinity. Taking into account the thickness of the surface in the form of a lamina allows one to analyze the fracture criteria for brittle materials without additional assumptions.

**2. Criteria for Crack Propagation in Brittle Materials.** *Energy Criterion of Fracture.* The conditions for brittle fracture can be formulated depending on the formulation of the problem. The energy criterion follows from the condition that the energy spent by the external forces to lengthen the crack does not exceed the energy of formation of the free surface of the crack. In this case, the crack propagates owing to the decrease in the elastic energy of the solid body. The energy of RE formation can be written in the form

$$dA = 0.5S(t)\,d\varepsilon_y\,d\sigma^r = \pi\sigma^2[a_0 + h(1-t)]\,h\,(1-t)(\beta+1)^2\Big(3 + 2\frac{a_0}{h(1-t)}\Big)t^\beta\,dt\,\frac{t^\beta}{2E_0}\,dt, \qquad (2.1)$$

where $S(t) = \pi[a_0 + h(1-t)]h(1-t)$ is the area of the ellipse (the relaxation region) of a given RE;

$$d\varepsilon_y = \frac{\sigma(\beta+1)}{E_0}\Big(3 + 2\frac{a_0}{h(1-t)}\Big)t^\beta\,dt$$

is the strain caused by relaxation, which is the same at all the points of the region of RE relaxation, and $E_0$ is the Young modulus.

We obtain the total energy spent for the formation of a crack with half-length $a_0$ by integrating expressions (2.1) twice over $t$ from 0 to 1. For $h \ll a_0$, we have $A = \pi\sigma^2 a_0^2/E$. The energy spent for the formation of the free surface of the crack is $A_0 = 4a_0\gamma$. The release of the elastic energy equals $\Delta A = A - A_0$.

If the lengthening of the crack is accompanied by the release of the elastic energy, the condition $d\Delta A/da \geqslant 0$ is satisfied. The critical fracture stress (the initiation of a crack) is found from

$$\sigma_{cr} \approx \sqrt{2\gamma E/\pi a_0}, \qquad (2.2)$$

and it coincides with the critical stress of the crack initiation according to Griffith–Orowan [2–5, 11].

*Strength Criterion of Fracture.* Our model makes it possible to formulate the strength criterion for the initiation of a crack as follows: a body with a crack of definite length fractures at a certain critical stress $\sigma_{cr}$ if the stress concentration in the neighborhood of the crack exceeds the value of the theoretical tensile strength, i.e., under the condition that $\sigma_{y\,\max} = \sigma_{cr}k_I = \sigma_{\text{theor}}$, where $k_I$ is the concentration coefficient. According to Eq. (1.5), for a crack of a given length the stress concentration is determined by the parameters $h$ and $\beta$. We mentioned above that the increase in $\beta$ decreases the physical width of the surface $h$ (see Fig. 2). Hence, it is meaningful to fix a definite value of the parameter $\beta$ for a specified width $h$ of the physical surface. For $\beta = 0$, the gradient of the Young modulus varies jumpwise at the boundary of contact between the surface layer and the elastically deformable matrix. When $\beta > 1$, the Young modulus varies smoothly in the range of the physical width of the surface $h$. However, the effective width of the surface markedly decreases in this case. Therefore, we take $\beta = 1$, which satisfies the condition of a smooth increase in the Young modulus in

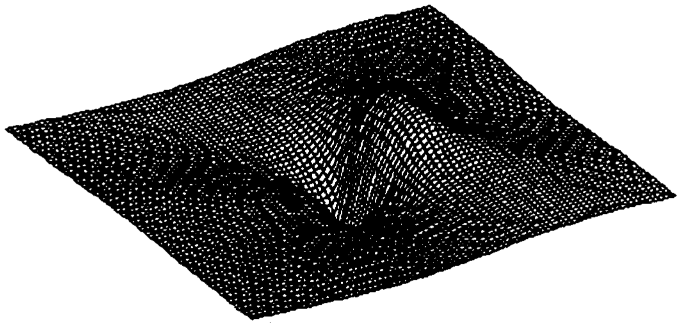

Fig. 3

the range of $h$ and is convenient to simplify the calculation of $\sigma_{y\,\max}$. For $a_0 \gg h$ and $\beta = 1$, the distribution of the stress $\sigma_y$ along the $x$ axis is expected to be determined by the formula

$$\sigma_y = \sigma_{\rm cr}\sqrt{\frac{a_0(x-a_0)}{2h^2}}\left(2 - \frac{x-a_0}{h}\right). \tag{2.3}$$

Taking the derivative $\partial\sigma_y/\partial x$ and equating it to zero, we find that the maximum of the stress $\sigma_y$ corresponds to $(x - a_0)/h = 2/3$. Substituting this value in Eq. (2.3), with allowance for $\sigma_{\rm cr}$ we obtain the theoretical value of the fracture strength of the material for $h = 3d$:

$$\sigma_{\rm theor} \approx 0.355\sqrt{\gamma E/\delta}.$$

Here $\delta$ is the interatomic distance. For example, for aluminum, at $\delta = 4 \cdot 10^{-10}$ m [11], $E = 72$ GPa [12], and $\gamma = 1.2$ J/m $^2$ [10, p. 14], we have $\sigma_{\rm theor} = 5.21$ GPa $\approx E/13$, which lies in the range of quantities predicted by known theoretical models [4–6, 13]. It is of interest to note that the values are quite reasonable for the physical thickness of the surface in the range of several interatomic distances despite the fact that our explicitly atomic structure of the surface layer has the property of a continuous medium with a variable Young modulus.

Thus, the strength and energy criteria are adequate in this model.

*Internal-Stress Field of a Crack.* We assume that the free surface of a crack is preserved after the external loading is removed. This means that there will be a definite internal-stress field in the volume of a solid body whose elastic energy is equal to work of the external forces spent for the formation of the surface layer. This field is due to the irreversibility of additional displacements in the near-surface layer at load.

It is evident that during unloading the free surfaces of the crack begin to come in contact along the boundary of the cut. The contact area gradually increases, beginning from the crack tip. It follows from the common considerations that after the complete unloading the compressive stress will act along the line of contact, which increases toward the crack tip. In the neighborhood of the tip, the material is exposed to the tensile force close to the theoretical strength and the corresponding strains. If these strains in the layer are considered irreversible, it follows from the equilibrium condition for the forces that upon unloading almost the same compressive stress will occur in this region.

If the displacements of the points of the surface layer, called a gradient of the effective moduli of elasticity, are irreversible, the internal-stress field of the crack will be adequate to the stress for a rectilinear band of localized plastic deformation [7, 8] with a distribution satisfying the smooth-change condition in the effective Young modulus from zero on the band axes to the value of $E_0$ at the boundary with the elastically deformable matrix. Figure 3 shows the spatial distribution of the internal stresses $\sigma_y$ of this field for $\beta = 1$. It is seen that the field is unperturbed only at the crack tip in the local area whose width is comparable with the width of the surface layer. For $\sigma_y$, in the coordinate system with origin at the crack tip we obtain the

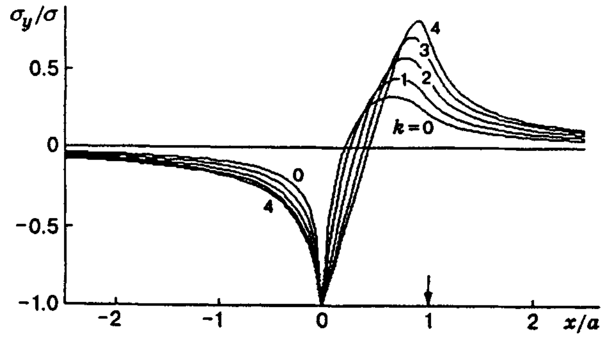


Fig. 4

expression

$$\sigma_y = 0.5\sigma(\beta+1)\int\limits_A^1 \frac{h(x-hz)z^{\beta+1}}{(x-hz)^2+y^2}\left[1+\frac{h^2z^2+4y^2}{(x-hz)^2+y^2}-\frac{4a^2y^2}{((x-hz)^2+y^2)^2}\right]dz+\sigma(\beta+1)\int\limits_0^A z^\beta\left(\frac{x}{hz}-1\right)dz.$$

In the neighborhood of the crack tip inside the circular region, i.e., under the condition that $(x-h)^2+y^2 \leqslant h^2$, we have $A=(x^2+y^2)/2xh$. Outside this region, only the first integral with $A=0$ acts.

The distribution of the stresses $\sigma_y$ along the axis of a semi-infinite crack is shown in Fig. 4. The material undergoes a compressive stress along the geometrical line of the crack, and its maximum is at the end of the cut. With distance from the end, this stress decreases rapidly and asymptotically tends to zero. This situation is close to the case where the length of the crack considerably exceeds the thickness of the surface layer. The microcracks in brittle materials, which are observed with an optical microscope, satisfy this condition. Since the stresses far from the tip of the crack are close to zero, the crack opening begins almost from the onset of loading. At a definite degree of opening, the stress concentration in the neighborhood of the tip becomes critical. Hence, the moment of crack initiation can also be characterized by a geometrical criterion, namely, the critical magnitude of the crack opening. It is evident that this criterion is unambiguously related to the energy (strength) criterion for crack propagation.

We shall present a more realistic situation from the physical viewpoint when the variation in the shape of the surface layer is reversible. In this situation, the intense compressive stresses upon unloading are expected to aim to close the crack according to the mechanism of "lightning." Experiments show that some microcracks in brittle materials are closed upon unloading [14]. For the crack to remain open in a solid body after unloading, the contour of the contact surfaces should be changed irreversibly. Generally, any local fracture of the material is preceded by plastic deformation of definite degree. Experience shows that, in practice, there are no absolutely brittle materials. Taking into account the plastic deformation considerably changes the stress distribution in the neighborhood of the crack. We shall analyze this effect.

**3. Effect of the Plastic Deformation on the Stress Concentration in a Cracked Solid.** We consider a simple case where the crack is surrounded by a plastically deformable material of elliptical shape. It is convenient to construct a site with a smooth field of plastic deformation by the method of relaxation elements [7–9]. Here the geometrical dimensions of the plastic region and the character of the plastic-deformation distribution in it is easily varied. The plastic-deformation site around the crack is shown schematically in Fig. 5. The plastic-deformation distribution ensuring the complete relaxation (the disappearance) of the stress inside the elliptical region bounded by the crack contour is first constructed with allowance for the physical width of the surface for the external stress $\sigma_1$ (the bold curve in Fig. 5). The corresponding stress field is found for this distribution of the ER. Then the solution for the crack (1.5) is imposed on this field with the external stress $\sigma_2$ so that the contour of the surface layer of the crack coincides with the boundary of the complete-relaxation region in the plastic-deformation site. As a result, we find the stress field of the crack with a plastic zone at the external stress $\sigma=\sigma_1+\sigma_2$.

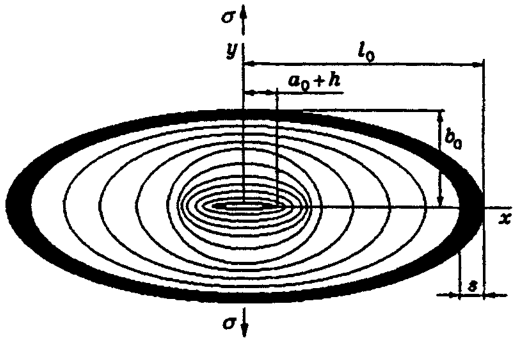


Fig. 5

The field of the plastic-deformation site is constructed using the RE similarly as was done for the crack with a near-surface layer. We specify the semi-axes of the RE connected to the plastic-deformation field: these are the semi-axes $l = a_0 + h + (l_0 - a_0 - h)(1-t)$ along the $x$ axis and $b = h + (b_0 - h)(1-t)$ along the $y$ axis. As before, $a_0 + h$ is the length of the crack with allowance for the physical width $h$, $l_0$ is the length of the plastic zone along the $x$ axis, and $b_0$ is the width of the plastic zone along the axis of tension. Inside each RE, we specify the magnitude of the elementary-relaxation tensor $d\sigma^r = \sigma_1(1-t)^\gamma\,dt$ for the RE in the unhatched region of the plastic-deformation site and the magnitude $d\sigma^r = \sigma_1 t^\gamma\,dt$ for other RE whose contours enter the near-boundary region of this site of width $s$ in the direction of the $x$ axis (see Fig. 5). An arbitrary RE in a given distribution creates an elementary stress field outside its relaxation region. According to [8], the distribution along the $x$ axis of the component $d\sigma_y$ of this field is described by the expression

$$d\sigma_y = \frac{l}{l-b}\left[\frac{b^2}{l(l-b)} + \frac{x(l-2b)}{(l-b)c} + \frac{b^2 x}{c^3}\right]d\sigma^r,$$

where $c = \sqrt{x^2 - (l^2 - b^2)(1-t)^2}$. It is connected with the existence of a homogeneous elementary field of plastic deformation $d\varepsilon_y^p = d\sigma^r(1 + 2l/b)$ inside the relaxation region of this RE.

For $l_0 \gg a_0$ and $b_0 \gg h$, one can assume that the ratio of the semi-axes of all the RE is the same, i.e., it equals the ratio of the semi-axes of the plastic-deformation site $l_0/b_0$. For the component $\sigma_y$ of the stress fields in this plastic-deformation site, the final result is as follows:

$$\frac{\sigma_y}{\sigma_1} = F(x) = \left[F_1(x)\left(\frac{l_0}{l_0 - s}\right)^{\gamma+1} + F_2(x)\left(\frac{l_0}{s}\right)^{\gamma+1}\right]. \tag{3.1}$$

Here

$$F_1(x) = \frac{b_0^2 A^{\gamma+1}}{(l_0 - b_0)^2} + \frac{b_0^2 x l_0(\gamma+1)}{l_0 - b_0}\left[\int\limits_{1-A}^{1} \frac{(1-t)^\beta}{c}\left(\frac{l_0 - 2b_0}{(l_0 - b_0)b_0^2} - \frac{(1-t)^2}{c^2}\right)dt\right],$$

where

$$A = \begin{cases} 1, & x \geqslant l_0 - s, \\ x/l_0, & x \leqslant l_0 - s; \end{cases}$$

$$F_2(x) = \frac{b_0^2[(s/l_0)^{\gamma+1} - B^{\gamma+1}]}{(l_0 - b_0)^2} + \frac{b_0^2 x l_0(\gamma+1)}{l_0 - b_0}\left[\int\limits_{B}^{s/l_0} \frac{t^\beta}{c}\left(\frac{l_0 - 2b_0}{(l_0 - b_0)b_0^2} - \frac{(1-t)^2}{c^2}\right)dt\right];$$

$$B = \begin{cases} 0, & x \geqslant l_0, \\ 1 - x/l_0, & x \leqslant l_0. \end{cases}$$

For $\gamma \geqslant 1$, $b_0 \gg h$, and $l_0 \gg a_0 + h$, i.e., when the dimensions of the plastic zone considerably exceed the size of the crack, the stresses are almost zero in the neighborhood of the center of the site. Therefore, one can add the stress field of the crack (1.5) to this solution if the external stress is $\sigma_2$.

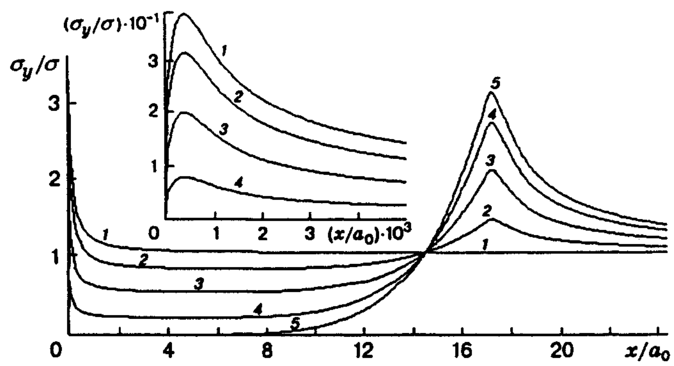


Fig. 6

The maximum plastic deformation near the crack is equal to $\varepsilon_y^p = \sigma_1(1+2l_0/b_0)/E$, i.e., it is determined by the quantity $\sigma_1$. Since the external stress is $\sigma = \sigma_1+\sigma_2$, the increase in $\sigma_1$ for the given stress $\sigma$ automatically decreases the stress near the crack.

Thus, after appropriate substitutions the full solution is written as the sum $\sigma_y = \sigma_1 F + \sigma_2 H$, where the function $F$ is determined by expression (3.1), and $H$ by expression (1.5) for the crack.

Figure 6 shows the distribution diagrams for the stress $\sigma_y$ along the $x$ axis in a solid body with a crack in the plastic zone for $\sigma_y/\sigma = 0$, 0.2, 0.5, 0.8, and 1 (curves 1–5, respectively). The distribution of the stress $\sigma_y$ at the crack tip is shown on a large scale. As expected, the accumulation of the plastic deformation decreases the stress concentration at the crack tip. The calculations show that, simultaneously, the stress concentration increases at the end of the plastic zone. However, the maximum stress concentration in the plastic-deformation zone is much less than the maximum stress concentration at the crack tip without the plastic-deformation zone. In our example, it differs by more than an order of magnitude. The increase in the concentration at the crack tip remains significant up to the high degrees of plastic deformation (curve 4). It is noteworthy that the gradients of the stress field at the tip are several orders of magnitude greater than those at the end of the plastic zone. These features clearly demonstrate the differences between the microconcentrator at the crack tip and the concentrator of a higher scale level, arising in the plastic-deformation zone.

The lengthening of the plastic zone does not affect the stress concentration at the crack tip, with other things being equal. However, the concentration at the end of the plastic-deformation zone depends on the change of the parameters $\gamma$ and $s$ in Eqs. (3.1). Since this concerns the property of the plastic-deformation zone rather than the crack, we omitted this dependence.

In concluding, it is noteworthy that the situation can arise where the plastic deformation completely suppresses the concentrator at the crack tip (curve 5 in Fig. 6).

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